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ON THE SECOND VARIATION, JACOBI'S EQUATION AND JACOBI'S THEOREM FOR THE INTEGRAL.

$$\int F(x, y, x', y')dt.$$

By A. Dresden.

Introduction.

In Weierstrass's theory of the integral

$$\int_{t_1}^{t_2} F(x, y, x', y')dt, \tag{1}$$

it is shown* that the second variation of (1) may be reduced to the form

$$\epsilon^2 \int_{t_1}^{t_2} (F_1 w'^2 + F_2 w^2)dt,$$

which is analogous to the form of the second variation of the integral

$$\int f\left(x, y, \frac{dy}{dx}\right) dx.$$

From this reduction of the second variation it follows that Jacobi's equation takes the form

$$F_2 u - (F_1 u')' = 0. \tag{2}$$

Then it is shown further, that when

$$x = x(t, \alpha), \qquad y = y(t, \alpha)$$

represent any one-parameter family of extremals for the integral (1), then

$$u = y'x_\alpha - x'y_\alpha$$

is a particular solution of equation (2).† This is Jacobi's theorem.

The purpose of this note is to obtain these results by means of a reduction of differential forms.‡ It appears to the author that in this way the derivation of the results becomes unified and less artificial.

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* See, e. g., Bolza, Vorlesungen über Variationsrechnung, § 28, 29. Notations and terms used without explanation in this note are taken in the sense given them in that book.

† This statement of Jacobi's theorem is slightly more general than what is usually given; it is adopted so as to include such extensions as are made, for instance, in my paper on "The second derivatives of the extremal integral," Transactions of the American Mathematical Society, vol. 9, p. 471.

‡ See *Annals of Mathematics*, vol. 13 (second series), p. 149.

We assume that the function F in (1) satisfies all the conditions of the classical theory and that we have found an extremal represented by

$$x = x(t), \qquad y = y(t), \qquad t_1 \leqq t \leqq t_2, \tag{3}$$

in which x and y are functions of class C''', which satisfy the initial conditions,

$$x(t_1) = x_1, \qquad y(t_1) = y_1, \qquad x(t_2) = x_2, \qquad y\,(t_2) = y_2,$$

and such that

$$[x'(t)]^2 + [y'(t)]^2 \neq 0, \qquad t_1 \leqq t \leqq t_2. \tag{3a}$$

If now ξ and η are two arbitrary functions of t of class C'' on (t_1t_2) and vanishing at t_1 and at t_2, the second variation of the integral (1) may be put in the form

$$\delta^2 I = \epsilon^2 \int_{t_1}^{t_2} 2\Omega dt,$$

where

$$2\Omega \equiv F_{xx}\xi^2 + 2F_{xy}\xi\eta + F_{yy}\eta^2 + 2F_{xx'}\xi\xi' + 2F_{xy'}\xi\eta' + 2F_{x'y}\xi'\eta$$
$$+ 2F_{yy'}\eta\eta' + F_{x'x'}\xi'^2 + 2F_{x'y'}\xi'\eta' + F_{y'y'}\eta'^2,$$

the arguments of the partial derivatives of F being

$$x = x(t), \qquad y = y(t), \qquad x' = x'(t), \qquad y' = y'(t).$$

By means of Euler's theorem on homogeneous functions this reduces to

$$\delta^2 I \equiv \epsilon^2 \int_{t_1}^{t_2} \left(\xi \frac{\partial \Omega}{\partial \xi} + \eta \frac{\partial \Omega}{\partial \eta} + \xi' \frac{\partial \Omega}{\partial \xi'} + \eta' \frac{\partial \Omega}{\partial \eta'} \right) dt, \tag{4}$$

in which expression the third and fourth terms may be integrated by parts, leading to

$$\partial^2 I \equiv \epsilon^2 \int_{t_1}^{t_2} (\xi\psi_1 + \eta\psi_2) dt,$$

where

$$\psi_1 \equiv \frac{\partial \Omega}{\partial \xi} - \frac{d}{dt}\frac{\partial \Omega}{\partial \xi'}, \qquad \psi_2 \equiv \frac{\partial \Omega}{\partial \eta} - \frac{d}{dt}\frac{\partial \Omega}{\partial \eta'}.$$

It is clear that ψ_1 and ψ_2 are linear differential forms of the second order in ξ and η, which satisfy the relation

$$x'\psi_1 + y'\psi_2 = 0. \tag{5}$$

Further, if

$$x = x(t, \alpha), \qquad y = y(t, \alpha)$$

represent a one-parameter family of extremals, which for $\alpha = \alpha_0$ contains the particular extremal (3), which is under consideration, it follows upon direct substitution,* that

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* As in the x-problem, see Bolza, l. c., p. 74.

$$\xi = x_\alpha(t, \alpha_0), \qquad \eta = y_\alpha(t, \alpha_0)$$

is a solution of the differential equations

$$\psi_1 = 0, \qquad \psi_2 = 0. \tag{6}$$

These equations may therefore be looked upon as the analogues of Jacobi's equation in the x-problem.

To the differential forms

$$\begin{aligned}\psi_1 \equiv &- F_{x'x'}\xi'' - F_{x'y'}\eta'' - F'_{x'x'}\xi' + (F_{xy'} - F_{x'y} - F'_{x'y'})\eta' \\ &+ (F_{xx} - F'_{xx'})\xi + (F'_{xy} - F'_{x'y})\eta, \\ \psi_2 \equiv &- F_{x'y'}\xi'' - F_{y'y'}\eta'' + (F_{x'y} - F_{xy'} - F'_{x'y'})\xi' - F'_{y'y'}\eta' \\ &+ (F_{xy} - F'_{xy'})\xi + (F_{yy} - F'_{yy'})\eta,\end{aligned}$$

we now apply the tests for reducibility,* which in this case take the form:

$$\begin{aligned}&\begin{vmatrix} F_{x'x'} & F_{x'y'} \\ F'_{x'x'} & F'_{x'y'} + F_{xy'} - F_{x'y} \end{vmatrix} = 0, \qquad \begin{vmatrix} F_{x'x'} & F_{x'y'} \\ F'_{xx'} - F_{xx} & F'_{xy'} - F_{xy} \end{vmatrix} = 0, \\ &\begin{vmatrix} F_{x'y'} & F_{y'y'} \\ F'_{x'y'} - F_{xy'} + F_{x'y} & F'_{y'y'} \end{vmatrix} = 0, \qquad \begin{vmatrix} F_{x'y'} & F_{y'y'} \\ F'_{x'y} - F_{xy} & F'_{y'y} - F_{yy} \end{vmatrix} = 0.\end{aligned} \tag{7}$$

To prove the first and third of these relations, we express every element of each of these determinants in terms of F_1 by means of formulæ, current in the theory.† To prove the second and fourth, we differentiate with respect to t the following formulae:‡

$$F_x \equiv x'F_{x'x} + y'F_{y'x}, \qquad F_y \equiv x'F_{x'y} + y'F_{y'y};$$

and, with respect to x' and y', the formula:§

$$F \equiv x'F_{x'} + y'F_{y'}.$$

This gives us:

$$\begin{cases} x'(F'_{x'x} - F_{xx}) + y'(F'_{y'x} - F_{xy}) = 0, \\ x'F_{x'x'} \qquad\qquad + y'F_{y'x'} \qquad\qquad = 0, \end{cases} \tag{8}$$

and

$$\begin{cases} x'(F'_{x'y} - F_{xy}) + y'(F_{y'y} - F_{yy}) = 0, \\ x'F_{x'y'} \qquad\qquad + y'F_{y'y'} \qquad\qquad = 0, \end{cases} \tag{9}$$

from which the desired relations follow, in view of condition 3(a).

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* See *Annals of Mathematics*, vol. 13, p. 152.

† See Bolza, l. c., p. 196 (12a), p. 203 (I).

‡ Bolza, l. c., p. 196 (11).

§ Ibidem, (10).

Hence we conclude that ψ_1 and ψ_2 are reducible to the form

$$\psi_i \equiv A_i w'' + B_i w' + C_i w, \qquad (i = 1, 2)$$

where w is any linear function of ξ and η, $w = a\xi + b\eta$, such that

$$\begin{vmatrix} a & b \\ F_{x'y'} & F_{y'y'} \end{vmatrix} = 0.*$$

Hence, we may put

$$w = -y'\xi + x'\eta,$$

and we can then determine A_i, B_i and C_i by undetermined coefficients. We find:

$$\begin{cases} y'A_1 = y'^2 F_1, \\ x'A_1 = x'y'F_1, \end{cases} \qquad \therefore\ A_1 = y'F_1,$$

$$\begin{cases} y'A_2 = -x'y'F_1, \\ x'A_2 = -x'^2 F_1, \end{cases} \qquad \therefore\ A_2 = -x'F_1,$$

$$\begin{cases} y'B_1 = y'^2 F_1', \\ x'B_1 = x'y'F_1', \end{cases} \qquad \therefore\ B_1 = y'F_1',$$

$$\begin{cases} y'B_2 = -x'y'F_1', \\ x'B_2 = -x'^2 F_1', \end{cases} \qquad \therefore\ B_2 = -x'F_1',$$

$$\begin{cases} x'C_1 = F_{xy} - F'_{x'y} - x'''y'F_1 - x''y'F_1', \\ y'C_1 = -F_{xx} + F'_{xx'} - y'''y'F_1 - y''y'F_1', \end{cases}$$

$$\begin{cases} y'C_2 = -F_{xy} + F'_{xy'} + y'''x'F_1 + y''x'F_1', \\ x'C_2 = F_{yy} - F'_{yy'} + x'''x'F_1 + x''x'F_1', \end{cases}$$

From Euler's differential equation for the integral (1), follows upon differentiation with respect to t:

$$F'_{x'y} - F'_{xy'} + y'(x''F_1)' - x'(y''F_1)' = 0,$$

and hence

$$F_{xy} - F'_{x'y} - x'''y'F_1 - x''y'F_1' \equiv F_{xy} - F'_{xy'} - y'''x'F_1 - y''x'F_1'.$$

By means of the same relation and formulæ (8_1) and (9_1) we verify:

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[*] See *Annals*, l. c., p. 153. We notice in passing that $w(t_1) = w(t_2) = 0$ for all choices of a and b.

$$\left\{\begin{aligned} & x'(+F'_{xx'} - F_{xx} - y'''y'F_1 - y''y'F_1') \\ & \qquad\qquad + y'(+F'_{x'y} - F_{xy} + x'''y'F_1 + x''y'F_1') = 0, \\ & x'(F'_{xy'} - F_{xy} + y'''x'F_1 + y''x'F_1') \\ & \qquad\qquad + y'(F'_{yy'} - F_{yy} - x'''x'F_1 - x''x'F_1') = 0. \end{aligned}\right.$$

Hence there exists a function F_2,* such that

$$F'_{xx'} - F_{xx} - y'''y'F_1 - y''y'F_1' = -y'^2F_2,$$

$$F'_{x'y} - F_{xy} + x'''y'F_1 + x''y'F_1' = F'_{xy'} - F_{xy} + y'''x'F_1 + y''x'F_1' = +x'y'F_2,$$

$$F'_{yy'} - F_{yy} - x'''x'F_1 - x''x'F_1' = -x'^2F_2.$$

Consequently, we have:

$$C_1 = -y'F_2,$$

and

$$C_2 = +x'F_2.$$

We can now write

$$\left\{\begin{aligned} \psi_1 &\equiv y'(F_1w'' + F_1'w' - F_2w), \\ \psi_2 &\equiv -x'(F_1w'' + F_1'w' - F_2w). \end{aligned}\right.$$

From this result, we conclude:

1. The second variation of the integral (1) may be reduced to the form:

$$\delta^2 I \equiv \epsilon^2 \int_{t_1}^{t_2} (y'\xi - x'\eta)(F_1w'' + F_1'w' - F_2w)dt.$$

Since w vanishes at t_1 and t_2, we may add to this the integral

$$\epsilon^2 \int_{t_1}^{t_2} \frac{d}{dt}(ww'F_1)dt \equiv \epsilon^2 \int_{t_1}^{t_2} (ww'F_1' + ww''F_1 + w'^2F_1)dt,$$

which gives us the expression:

$$\delta^2 I \equiv \epsilon^2 \int_{t_1}^{t_2} (F_2w^2 + F_1w'^2)dt,$$

known from the current theory.

2. Jacobi's differential equation for the integral (1) reduces to

$$F_1w'' + F_1'w' - F_2w = 0, \tag{2}$$

since

$$x'^2 + y'^2 \neq 0,$$

which is known as Weierstrass's form of Jacobi's equation.

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* It can be verified without much difficulty that the function F_2 here defined is identical with the function introduced by Weierstrass.

3. Since the functions

$$\xi = x_\alpha(t, \alpha_0), \qquad \eta = y_\alpha(t, \alpha_0)$$

satisfy equations (6), it follows that the function:

$$w = x'y_\alpha - y'x_\alpha$$

will satisfy the Weierstrass form of this equation,

$$x = x(t, \alpha), \qquad y = y(t, \alpha)$$

representing any one-parameter family of extremals: Weierstrass's form of Jacobi's theorem.

The extension of this method to the treatment of the second variation in the more general problems of the calculus of variations, which seems to be possible, has not as yet been obtained.

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